The Black-Scholes-Merton Model

Part I. Brownian Motion. Reference: Hull, Chapter 12
A Discrete-time Prototype for Brownian Motion

Suppose we have a finite time interval $[0, T]$. We divide it into N time periods of length Δt each and equally spaced by $N + 1$ time points $t_i = i\Delta t$, $i = 0, 1, 2, ..., N$. Consider a discrete-time stochastic process defined as follows:

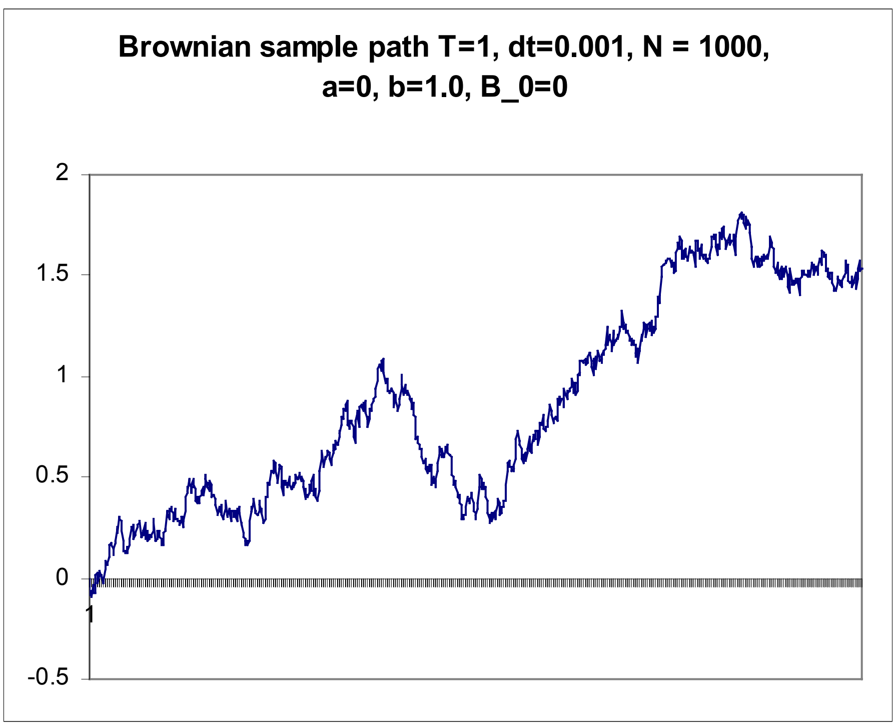
$B_0 = 0,$

$B_{i+1} = B_i + \varepsilon_{i+1}\sqrt{\Delta t}$, $i = 0, 1, ..., N-1,$

where $\{\varepsilon_i, i = 1, 2, ..., N\}$ are independent standard normal random variables, $\varepsilon_i \sim N(0,1)$, $i = 1, 2, ..., N$.

Monte Carlo Simulation

The figure depicts a *sample path* (particular realization) of the process with $T = 1$ and $\Delta t = 0.001$ ($N = 1{,}000$). This sample path is produced by Monte Carlo simulation in *Excel* using *Analysis ToolPack*. The process is starting at zero, $B_0 = 0$. At the next time step t_1, I draw a sample ε_1 from the standard normal distribution using the *Random Number Generation Tool* in *Excel* and calculate the value of the process $B_1 = B_0 + \varepsilon_1\sqrt{\Delta t}$. I continue in this fashion until I have simulated the entire sample path. The final simulated value is $B_N = B_{N-1} + \varepsilon_N\sqrt{\Delta t}$.



Properties of the process $\{B_i = B(t_i),\ \ i = 0,1,...,N\}$:

1. Since $B_i = \left(\sum_{k=1}^{i} \varepsilon_k\right)\sqrt{\Delta t}$ and $\sum_{k=1}^{i} \varepsilon_k$ is the sum of *i* independent standard normal random variables and thus is normally distributed with zero mean and variance equal to *i*, B_i is normally distributed with zero mean and variance equal to $i\Delta t = t_i$ (time elapsed from the start at $t = 0$ to the current time t_i):

$$E[B_i] = 0,$$

$$\mathrm{Var}[B_i] = E[B_i^2] = t_i.$$

2. The increment of the process

$$\Delta B_i := B_{i+1} - B_i = \varepsilon_{i+1}\sqrt{\Delta t}$$

is normally distributed with zero mean and variance equal to Δt.

3. More generally, the increment of the process $B_i - B_j$ is normally distributed with zero mean and variance

$$(i - j)\Delta t = t_i - t_j.$$

The variance of the increment $B_i - B_j$ is equal to the length of the time interval between t_i and t_j

Brownian Motion

A *standard Brownian motion process* (or *Wiener process*) $\{B_t = B(t), 0 \le t \le T\}$ is obtained by taking the limit $\Delta t \to 0$. In the limit of infinitesimal time steps *dt*, *formal* properties of infinitesimal increments of standard Brownian motion are:

$$dB_t = \varepsilon\sqrt{dt}\ , \quad \varepsilon \sim N(0,1),$$

$$E[dB_t] = 0\ ,$$

$$\mathrm{Var}[dB_t] = E[dB_t^2] = dt\ .$$

Recall that the variance of $(\Delta B)^2$ is of the order Δt^2. In the limit we can approximate it as non-random and equal to its expected value:

$$(dB_t)^2 = dt\ .$$

The Wiener process can be mathematically defined by listing its defining properties.

Definition A real-valued stochastic process $\{B(t), t \ge 0\}$ with continuous sample paths is a *standard Brownian motion* (*Wiener process*) if it satisfies the following properties:

1. The process is starting at the origin:

$$B(0) = 0\ .$$

2. For any $0 \le t_1 < t_2$, the increment $B(t_2) - B(t_1)$ is a normal random variable with zero mean and variance equal to $t_2 - t_1$:

$$E[B(t_2) - B(t_1)] = 0\ ,$$

$$\mathrm{Var}[B(t_2) - B(t_1)] = E[(B(t_2) - B(t_1))^2] = t_2 - t_1\ .$$

3. For any $0 \le t_1 < t_2 \le t_3 < t_4$, the increments $B(t_2) - B(t_1)$ and $B(t_4) - B(t_3)$ are independent.

These properties parallel the properties of the discrete-time process we have constructed. Brownian sample paths are continuous but not differentiable with respect to time – they are very jagged! They also have unbounded variation.

Adding Drift: Brownian Motion with Drift Coefficient *a*, Diffusion Coefficient *b*, and Starting at *x*

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and $a, b, x \in \mathbb{R}$. Consider a process $\{X_t, t \geq 0\}$ defined by:

$$X_t = x + at + bB_t, \qquad t \geq 0.$$

This process is called Brownian motion (Wiener process) with drift coefficient *a*, diffusion coefficient *b*, and starting at *x*.

Properties:

1. The process is starting at some point *x* at time zero.

2. For any $0 \leq t_1 < t_2$, the increment $X(t_2) - X(t_1)$ is a normal random variable with mean $a(t_2 - t_1)$ and variance equal to $b^2(t_2 - t_1)$:

$$E[X(t_2) - X(t_1)] = a(t_2 - t_1),$$

$$\text{Var}[X(t_2) - X(t_1)] = E[(X(t_2) - X(t_1))^2] = b^2(t_2 - t_1).$$

3. For any $0 \leq t_1 < t_2 \leq t_3 < t_4$, the increments $X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are independent.

The parameter *a* measures the *drift rate* of the process (the slope of the deterministic trajectory at). The parameter *b* measures the *dispersion rate* of the process (the process *diffuses* around the deterministic trajectory at with diffusion coefficient *b*).

Monte Carlo Simulation

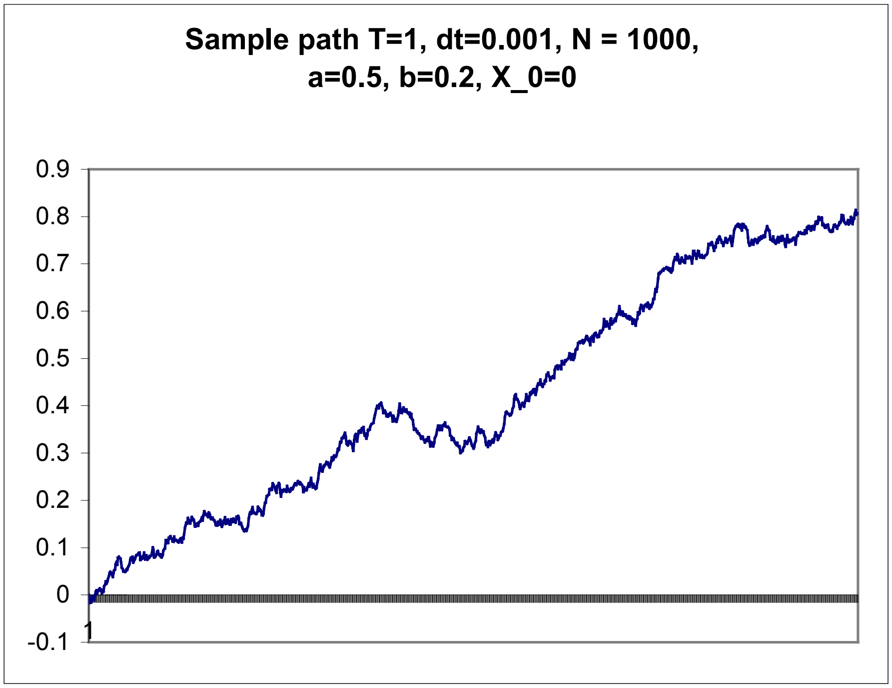
The following figure depicts a sample path of the process with *a* = 0.5 and *b* = 0.2 (*x* = 0). This figure is produced by approximating the continuous-time process:

$$X_0 = x,$$

$$X_{i+1} = X_i + a\Delta t + b\Delta B_i, \text{ or } \Delta X_i = a\Delta t + b\Delta B_i, \; i = 0, 1, \ldots, N-1,$$

$$\Delta B_i = \varepsilon_{i+1}\sqrt{\Delta t},$$

where I discretized the time interval $[0, T]$ into *N* discrete time steps Δt. This discrete-time process was simulated in Excel as was described previously. I used *T* = 1 and *N* = 1,000 to produce this graph ($\Delta t = 0.001$).



Diffusion Processes

Monte Carlo Simulation

The next step is to make drift and diffusion coefficients depend on the state variable and time:

$X_0 = x$,

$X_{i+1} = X_i + a(X_i, t_i)\Delta t + b(X_i, t_i)\Delta B_i$,or $\Delta X_i = a(X_i, t_i)\Delta t + b(X_i, t_i)\Delta B_i$, $i = 0, 1, ..., N-1$,

$\Delta B_i = \varepsilon_{i+1}\sqrt{\Delta t}$,

where $a(x,t)$ and $b(x,t)$ are some well-behaved functions. In the Black-Scholes-Merton model *a* and *b* are linear functions.

Continuous-time Limit

In the limit of Δt getting smaller and smaller, we obtain a continuous-time stochastic process called *Ito diffusion process*:

$X_0 = x$,
$dX_t = a(X_t, t)dt + b(X_t, t)dB_t$.
Drift and diffusion coefficients depend on the state variable and time.

Historic remarks

It is remarkable that detailed studies of Brownian motion were initiated at the turn of the 20^{th} century motivated by problems in finance. In 1900 Louis Bachelier in Paris, France showed in his thesis under Henri Poincare that Brownian motion process is at the heart of modeling asset prices. He modeled stock prices on the *Paris Bourse* (Paris Stock Exchange) by what is now known as Brownian motion or Wiener process. It took over 60 years until Paul Samuelson made the next step in the sixties and proposed a model for asset prices as Geometric Brownian motion. Following in Samuelson's footsteps, Robert Merton, Samuelson's student, initiated applications of stochastic calculus in finance and economics.

In 1905, and completely independently of Bachelier, Albert Einstein developed mathematical theory of the physical phenomenon of Brownian motion first observed by English botanist Robert Brown in 1827 and used the same mathematical tools as Bachelier. Norbert Wiener developed the first mathematically rigorous theory of Brownian motion as a stochastic process in 1923. Later Kolmogorov in Russia, Levy in France and Ito in Japan further developed this theory into the modern theory of stochastic processes.

Part II. Geometric Brownian Motion Model of Asset Price Dynamics

1. Geometric Brownian Motion with Drift as a Model for Asset Prices
2. Lognormal Distribution of Asset Prices and its Properties
3. Monte Carlo Simulation of Geometric Brownian Motion
4. Estimating Historical Volatility from Time-series Market Data

First, consider a riskless asset: a *money market account*. At time zero we deposit one dollar:

$$A_0 = 1 .$$

The money market account balance grows at the continuously compounded risk-free rate r. Over an infinitesimal time period dt the change in the account value is:

$$dA_t = rA_t dt .$$

The *percentage return* over dt is:

$$\frac{dA_t}{A_t} = rdt .$$

At time $t \geq 0$ the value of our money market account is:

$$A_t = e^{rt} .$$

Suppose now that S_t is the price at time t of a risky asset (e.g. stock) that pays no dividends (we will extend to the case with dividends later). We will model its time evolution by some Ito diffusion process. The stock price at time zero is known:

$S_0 = S$,

and the change in the stock price over an infinitesimal time period *dt* is modeled as an increment of the diffusion process:

$$dS_t = a(S_t, t)dt + b(S_t, t)dB_t .$$

A simple and natural choice for the drift and diffusion coefficients is:

$$dS_t = mS_t dt + \sigma S_t dB_t .$$

This process with linear drift $a(S,t) = mS$ and diffusion $b(S,t) = \sigma S$ is called *geometric Brownian motion*. Over an infinitesimal time period *dt* the percentage return on the stock is:

$$\frac{dS_t}{S_t} = mdt + \sigma dB_t .$$

Here *m* is the constant annualized instantaneous expected rate of return on the stock (drift rate),

$$E_t\left[\frac{dS_t}{S_t}\right] = mdt ,$$

and σ is the *volatility of the stock price*. The assumption of constant volatility means that the variance of the percentage return over an infinitesimal period of time *dt* is the same regardless of the stock price. That is, $\sigma^2 dt$ is the variance of the percentage return on the stock over *dt*, $\frac{dS_t}{S_t}$:

$$\text{Var}_t\left[\frac{dS_t}{S_t}\right] = \sigma^2 dt$$

(recall the properties of the Brownian increment $E[dB_t] = 0$ and $\text{Var}[dB_t] = dt$).

What is the distribution of S_T?

Consider a new process:

$$X_t = \ln S_t, \quad t \geq 0 .$$

Ito's lemma tells us that a function *f* of *S* follows an Ito process:

$$dX_t = \left(m - \frac{\sigma^2}{2} \right) dt + \sigma dB_t \, .$$

Hence, the logarithm of the stock price follows a Brownian motion with drift coefficient $m - \sigma^2 / 2$, diffusion coefficient σ and starting at $\ln S_0$:

$$X_t = \ln S_t = \ln S_0 + \nu t + \sigma B_t \, , \quad t \geq 0 \, ,$$

or

$$\ln\left(\frac{S_t}{S_0} \right) = \nu t + \sigma B_t \, , \; t \geq 0 \, ,$$

where B_t is a standard Brownian motion and $\nu := m - \dfrac{\sigma^2}{2}$.
Exponentiating we obtain:

$$S_t = S_0 e^{\nu t + \sigma B_t} \, , \; t \geq 0 \, .$$

The value of B_t at time $t > 0$ is normally distributed with zero mean and variance equal to t. Thus, the future stock price S_t at time $t > 0$, given the stock price S_0 at time zero, is distributed according to:

$$S_t = S_0 e^{\nu t + \varepsilon \sigma \sqrt{t}} \, ,$$

where ε is a standard normal random variable. This is the *lognormal*. We now find its pdf. First, observe that given the logarithm $\ln S_0$ of the stock price at time zero, $\ln S_t$ is normally distributed with mean $\ln S_0 + \nu t$ and variance $\sigma^2 t$.

Properties of the Lognormal Distribution

Recall that $S_t = S_0 e^{\eta_t}$, where $\eta_t = \nu t + \varepsilon\sigma\sqrt{t}$ is a normal random variable with mean νt and variance $\sigma^2 t$. Using the result that if X is a normal random variable with mean a and standard deviation b, then

$$E[e^X] = e^{a+b^2/2},$$

and recalling that $\nu = m - \sigma^2/2$, we obtain the moments:

$$E[S_t^n] = S_0^n E[e^{n\eta_t}] = S_0^n \exp\left\{ n\nu t + \frac{n^2\sigma^2 t}{2} \right\}$$

$$= S_0^n \exp\left\{ nmt + \frac{n(n-1)\sigma^2 t}{2} \right\}$$

The mean and variance of the lognormal distribution:

$$E[S_t] = e^{mt} S_0,$$

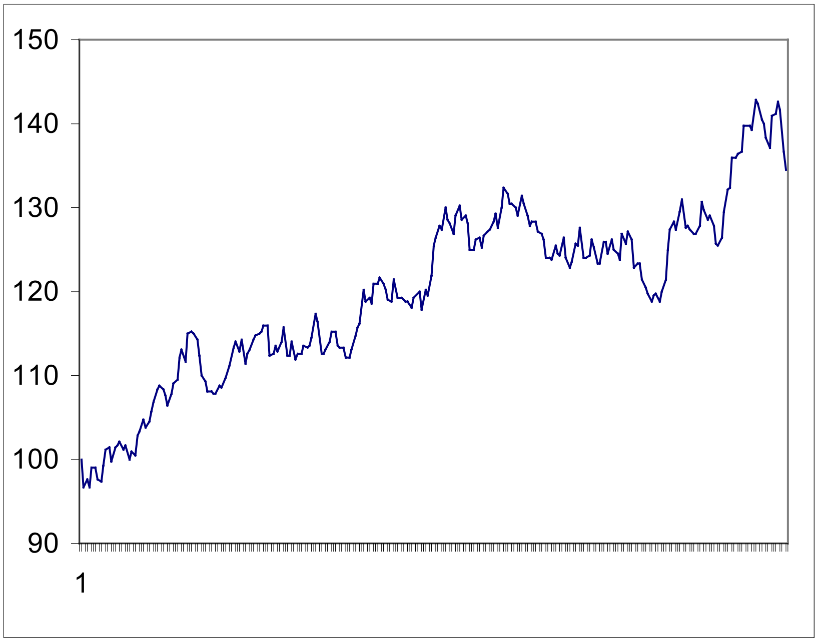
$$\mathrm{Var}[S_T] = S_0^2 (e^{2mt+\sigma^2 t} - e^{2mt}) = S_0^2 e^{2mt} (e^{\sigma^2 t} - 1).$$

Monte Carlo Simulation of Geometric Brownian Motion

Suppose we wish to simulate a sample path of the geometric Brownian motion process for the stock price. We divide the time interval $[0,T]$ into N equal time steps $\Delta t = \frac{T}{N}$ and simulate a sample path $\{S(t_i), i = 0, 1, \ldots, N\}$, $t_i = i\Delta t$, starting from the known price S_0 at time $t_0 = 0$:

$$S_{i+1} = S_i e^{\nu\Delta t + \sigma\varepsilon_{i+1}\sqrt{\Delta t}}, \ i = 0,1,\ldots,N-1,$$

where ε_{i+1} are independent samples from the standard normal distribution. The following figure depicts a sample path of the GBM process $T = 1$ year, $m = 15\%$ and $\sigma = 20\%$ per annum, initial stock price $S = 100$, $N = 250$.



For small Δt, if we expand

$$S_{i+1} = S_i e^{\nu\Delta t + \sigma\varepsilon_{i+1}\sqrt{\Delta t}}$$

in the Taylor series and drop the terms of orders higher than Δt, we obtain:

$$S_{i+1} = S_i\left(1 + \left(\nu + \sigma^2/2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_{i+1}\right)$$
$$= S_i + S_i m\Delta t + S_i\sigma\sqrt{\Delta t}\varepsilon_{i+1}$$

Note that the extra term with $\sigma^2\Delta t/2$ came from the square of the increment $\sigma^2(\Delta B_i)^2/2$, where $(\Delta B_i)^2$ is approximated as a deterministic quantity equal to its mean Δt.

Parameters *m* and σ

The expected percentage return on the stock over an infinitesimal time period *dt* is *mdt*, where *m* is the expected percentage rate of return (annualized continuously compounded percentage rate). Typical values for the broad stock market indexes have been around 9% - 10% over the last hundred years. The parameter σ is the *instantaneous volatility* defined as the annualized instantaneous standard deviation of the stock percentage return. That is, standard deviation of the percentage return on the stock over an infinitesimal time interval dt is $\sigma\sqrt{dt}$. Historical values have been around 14% to 20% per annum for the S&P 500 index and 10% to 50% for individual stocks.

Estimating Volatility from Historical Data

Suppose we observe stock prices at fixed small time intervals Δt (e.g., daily price observations) and suppose we have a total of $N + 1$ observations (e.g., daily closing prices):

$S_i = S(t_i)$: stock price at the end of ith interval,

$i = 0, 1, \ldots, N$,

T: total length of the observation interval (in years).

Our model states that

$$\ln\left(\frac{S_i}{S_{i-1}}\right) = \nu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i\,, \quad i = 1,2,\ldots,N\,,$$

where ε_i are standard normal random variables and $\nu = m - \frac{\sigma^2}{2}$.

Define u_i as the continuously compounded rate of return over the ith interval:

$$S_i = S_{i-1}e^{u_i} \qquad \text{or} \qquad u_i = \ln\left(\frac{S_i}{S_{i-1}}\right), \quad i = 1, 2, \ldots, N.$$

Our model says

$$u_i = \nu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i\,.$$

Then the estimate $\hat{\sigma}$ of annualized volatility σ can be obtained from (sample variance):

$$\hat{\sigma}^2\Delta t = \frac{1}{N-1}\sum_{i=1}^{N}(u_i - \overline{u})^2$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}u_i^2 - \frac{1}{N(N-1)}\left(\sum_{i=1}^{N}u_i\right)^2$$

where $\overline{u}$ is the sample mean of u_i , $\overline{u} = \frac{1}{N}\sum_{i=1}^{N}u_i$.

Note that $\hat{\sigma}^2\Delta t$ is the estimate of the variance per one time interval Δt (e.g., one trading day), and $\hat{\sigma}^2$ is the annualized variance estimate.

Part III. The Black-Scholes-Merton Model

Readings: Hull, Chapter 13

Derivation of the Black-Scholes-Merton Partial Differential Equation (PDE)

Consider a European-style derivative security with the payoff $F(S_T)$ at expiration at *T*.

- What is the price $f_t = f(S_t, t)$ of this security at some time $t < T$ when the underlying asset price is S_t?
- How do we hedge this security?

The derivation is similar to the derivation when the underlying asset follows the binomial process: we will *dynamically replicate* the security by trading the underlying asset.

Modeling Assumptions

1. The underlying asset follows geometric Brownian motion starting at $S_0 = S$:

$dS_t = mS_t dt + \sigma S_t dB_t$, where m and σ are constant;

2. No restrictions on short sales;
3. No transaction costs, bid/ask spread, or taxes;
4. No dividends during the lifetime of the derivative security;
5. No riskless arbitrage opportunities (or they are explored immediately as they occur);
6. Securities trading is continuous, securities are infinitely divisible, and there are no liquidity restrictions;
7. The continuously compounded risk-free rate r is constant and is the same for all maturities (flat term structure of interest rates).

Derivation of the Black-Scholes PDE: dynamic hedging strategy (hedging portfolio)

The price of our derivative at time t is a function of the underlying price S_t and time t. From Ito's lemma:

$$df(S_t,t) = \left(\frac{\partial f}{\partial t}(S_t,t) + mS_t \frac{\partial f}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}(S_t,t) \right) dt + \sigma S_t \frac{\partial f}{\partial S}(S_t,t) dB_t.$$

The key observation is that both the stock and the option have the same source of uncertainty (risk) – Brownian motion process B.

We construct a portfolio with a short position in one derivative f and a long position in Δ shares of stock:

$$\Pi = \Delta S - f\ .$$

From Ito's lemma, the change in the portfolio value over *dt* is given by (dropping time subscripts and function arguments to lighten notation):

$$d\Pi = \Delta dS - df = \Delta(mSdt + \sigma SdB)$$
$$-\left(\frac{\partial f}{\partial t} + mS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt - \sigma S\frac{\partial f}{\partial S}dB$$
$$= \sigma S\left(\Delta - \frac{\partial f}{\partial S}\right)dB + \left(mS\left(\Delta - \frac{\partial f}{\partial S}\right) - \frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt.$$

Let us select the number of shares (*hedge ratio* or *delta*) Δ so that

$$\Delta = \frac{\partial f}{\partial S}.$$

This selection makes our portfolio *instantaneously riskless* – the coefficient in front of the term with the Brownian motion increment *dB* (risk) vanishes!

Compare this choice with the choice

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{C_u - C_d}{(u-d)S}$$

in the binomial approach.

Our portfolio is riskless only for an infinitesimal time period *dt*. To keep our portfolio riskless through the next time period *dt*, we need to *re-balance* – adjust the delta to reflect the change in the stock price.

Since our portfolio is *instantaneously* riskless over an infinitesimal time period *dt*, its rate of return must be equal to the risk-free rate *r* (otherwise, there is an *arbitrage opportunity*)! Then the change in value of our portfolio during an infinitesimal time period *dt* must be:

$$d\Pi = r\Pi dt, \text{ or } \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt = r\left(S\frac{\partial f}{\partial S} - f\right)dt,$$

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf. \qquad (*)$$

The price of the option as a function of the underlying stock price and time $f = f(S,t)$ must satisfy this partial differential equation (PDE) as a consequence of the no arbitrage assumption. If the option price does not satisfy this differential equation, there are arbitrage opportunities.

This is the celebrated Black-Scholes-Merton PDE, also called the fundamental pricing equation. Mathematically, this has the form of diffusion equation. At expiration of the derivative $t = T$, the differential equation should be supplemented with the terminal condition at expiration (payoff):

$f(S,T) = F(S).$

To find derivative's price at any time *t* prior to expiration, $0 \le t < T$, we need to solve the Black-Scholes-Merton PDE subject to the terminal condition at time expiration *T*:

$f(S,T) = F(S).$

Note that we need to solve the PDE backwards in time (subject to the *terminal* condition rather than the *initial* condition) since we know the payoff and need to find the present value (current price) $f(S,t)$ of the security with this payoff:

←――――――――――――

$t, f(S,t)$ $\qquad\qquad$ $T,\ f(S_T,T) = F(S_T)$

For European Call Options
$F(S) = (S - K)^+$

For European Put Options
$F(S) = (K - S)^+$

Note that delta (hedge ratio) depends both on the underlying price and time:

$$\Delta = \Delta(S,t) = \frac{\partial f}{\partial S}(S,t),$$

where the function $f(S,t)$ is the solution of the Black-Scholes-Merton PDE. As *t* and *S* change, we need to *rebalance* our portfolio at each (infinitesimally small) time step *dt* → dynamic trading strategy.

Important! Note that the drift rate *m* does *not* enter the equation. The equation involves risk-free rate *r* and volatility σ as the only parameters. This is similar to the binomial model – recall that the probability *q* fell out of the equation! The drift rate of the stock in real world does not matter for pricing derivatives! For the purpose of pricing, we can pretend all investors are risk-neutral (*risk-neutral world*), set *m* = *r*, and use the risk-free rate *r* both as the drift rate of the stock and for discounting. Read Section 13.7 in Hull.

Stock price process in the risk-neutral world:

$$dS_t = rS_t dt + \sigma S_t dB_t ,$$

or

$$S_t = S_0 e^{\mu t + \sigma B_t} ,\ t \geq 0$$

where

$$\mu = r - \sigma^2 / 2 .$$

Deriving the Black-Scholes-Merton Option Pricing Formulas

One way to proceed is to find the option price by directly solving the PDE subject to the payoff condition.

Alternatively, we can calculate the option price (present value) at time $t < T$ when the stock price is S as the discounted expectation of its payoff in the *risk-neutral world*:

Risk-Neutral Pricing Formula

$$f(S,t) = e^{-r\tau} E_{t,S}[F(S_T)] , \qquad \tau = T - t ,$$

$$S_T = S e^{\mu\tau + \varepsilon\sigma\sqrt{\tau}} , \qquad \mu = r - \frac{\sigma^2}{2} .$$

This is the *risk-neutral valuation formula*. The risk-free rate r is used as the discount rate. In the risk-neutral world, given today's stock price S at time t, the future stock price S_T at time $T > t$ is lognormally distributed: $S_T = S e^{\mu\tau + \varepsilon\sigma\sqrt{\tau}}$, where ε is standard normal.

Calculating Risk-Neutral Expectations

Example 1: Forward Contracts

By applying the risk-neutral valuation formula to the payoff of a forward contract with delivery price K, $(S_T - K)$, we can find PV of the forward contract ($\tau = T - t$):

$$f(S,t) = e^{-r\tau} \left(E_{t,S}[S_T] - K \right) = S - e^{-r\tau} K ,$$

where we used the formula for the mean of the lognormal distribution:

$$E_{t,S}[S_T] = e^{r\tau} S .$$

Not surprisingly, this is the same formula we obtained previously from the no-arbitrage argument!

It is instructive to check that this value of the forward contract does indeed satisfy the BSM PDE: calculating all the necessary derivatives, the BS PDE reduces to:

$$rS - re^{-r\tau}K = r(S - e^{-r\tau}K)\,.$$

Black-Scholes-Merton Option Pricing Formulas

Example 2: European Call Option

Given the stock price S at time $t = 0$, in the risk-neutral world the stock price at time $T > t$ is a lognormal random variable:

$$S_T = Se^{\mu\tau+\sigma\sqrt{\tau}\varepsilon},$$

where $\tau = T - t$ is time to maturity, $\mu = r - \frac{\sigma^2}{2}$ and ε is standard normal. Then the call price is given by the discounted risk-neutral expectation:

$$C(S,t) = e^{-r\tau}E[(S_T - K)^+]$$
$$= e^{-r\tau}E[(Se^{\mu\tau+\varepsilon\sigma\sqrt{\tau}} - K)^+]$$
$$= e^{-r\tau}SE[e^{\mu\tau+\varepsilon\sigma\sqrt{\tau}}1_{\{\mu\tau+\varepsilon\sigma\sqrt{\tau}>\ln(K/S)\}}] - e^{-r\tau}KE[1_{\{\mu\tau+\varepsilon\sigma\sqrt{\tau}>\ln(K/S)\}}]$$
$$= e^{-\sigma^2\tau/2}SE[e^{\mu\tau+\varepsilon\sigma\sqrt{\tau}}1_{\{\varepsilon>(\ln(K/S)-\mu\tau)/(\sigma\sqrt{\tau})\}}]$$
$$-e^{-r\tau}KE[1_{\{\varepsilon>(\ln(K/S)-\mu\tau)/(\sigma\sqrt{\tau})\}}]$$

Introduce the following notation:

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + \mu\tau}{\sigma\sqrt{\tau}}$$

Then

$$C(S,t) = e^{-\sigma^2\tau/2}SE[e^{\varepsilon\sigma\sqrt{\tau}}1_{\{\varepsilon>-d_-\}}] - e^{-r\tau}KE[1_{\{\varepsilon>-d_-\}}]$$

Here $1_{\{\varepsilon>-d_-\}} = 1$ if $\varepsilon > -d_-$ and = 0 if $\varepsilon \le -d_-$.

The expectation in the second term:

$$E[1_{\{\varepsilon>-d_-\}}] = \Pr(\varepsilon > -d_-) = \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-\frac{z^2}{2}} dz = N(d_-)$$

where we changed the integration variable $z = -y$ and used the definition of the standard normal CDF:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy .$$

The expectation in the first term:

$$E[e^{\varepsilon\sigma\sqrt{\tau}} 1_{\{\varepsilon>-d_-\}}] = \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} e^{-\frac{y^2}{2}+y\sigma\sqrt{\tau}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} e^{-\frac{(y-\sigma\sqrt{\tau})^2}{2}+\frac{\sigma^2\tau}{2}} dy = e^{\frac{\sigma^2\tau}{2}} \int_{-\infty}^{d_-+\sigma\sqrt{\tau}} e^{-\frac{z^2}{2}} dz = e^{\frac{\sigma^2\tau}{2}} N(d_+)$$

where

$$d_+ = d_- + \sigma\sqrt{\tau} .$$

Substituting the results for both terms in the call pricing formula, we finally arrive at:

The Black-Scholes Call Pricing Formula

$$C(S,t) = SN(d_+) - e^{-r\tau} KN(d_-) ,$$

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + \mu\tau}{\sigma\sqrt{\tau}}, \qquad d_+ = d_- + \sigma\sqrt{\tau}$$

$$\tau = T - t , \qquad \mu = r - \frac{\sigma^2}{2} .$$

Financial meaning of the first term $SN(d_+)$ of the Black-Scholes formula: present value of receiving the stock at maturity, provided the call finishes in the money.
Financial meaning of the second term $-e^{-r\tau}KN(d_-)$: present value of paying the strike, provided the call finishes in the money.

It is easy to calculate the call delta by taking the first derivative of the option's price with respect to the underlying stock price:

$$\Delta = \frac{\partial C}{\partial S} = N(d_+) ,$$

where we have used the following identity (can be verified directly by differentiating):

$$S \frac{\partial N(d_+)}{\partial S} - e^{-r\tau} K \frac{\partial N(d_-)}{\partial S} = 0 .$$

Thus, we can also re-write the option pricing formula in the form:

$$C = \Delta S - B .$$

Compare this to the binomial pricing formula!

Example 3 European Puts.

A similar derivation yields the put pricing formula:

The Black-Scholes Put Pricing Formula

$$P(S,t) = e^{-r\tau} KN(-d_-) - SN(-d_+) .$$

It is now easy to verify the put-call parity:

$$C - P = S[N(d_+) + N(-d_+)] - e^{-r\tau} K[N(d_-) + N(-d_-)]$$
$$= S - e^{-r\tau} K$$

where we have used the identity

$$N(x) + N(-x) = 1 .$$

Options on Stock Indexes, Foreign Currencies, and Futures

The Black-Scholes formula allows us to price European options on the underlying asset that pays no dividends during the life of the option. To use the model in practice, we need to extend it to allow for dividends.

Suppose the underlying asset pays continuous proportional dividends with the constant dividend yield q per annum (with continuous compounding). Dividends paid out over an infinitesimal time period dt are $S_t q dt$.

The Black-Scholes European Option Pricing Formula adjusted for dividends

We can derive it by substituting $S \to e^{-q\tau} S$ into the BS formula without dividends:

$$C = e^{-q\tau} S N(d_+) - e^{-r\tau} K N(d_-),$$

$$P = e^{-r\tau} K N(-d_-) - e^{-q\tau} S N(-d_+),$$

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + \mu\tau}{\sigma\sqrt{\tau}}, \quad d_+ = d_- + \sigma\sqrt{\tau},$$

$$\mu = r - q - \frac{\sigma^2}{2}.$$

To get the adjustment in μ just note that

$$\ln\left(\frac{e^{-q\tau}S}{K}\right) = \ln\left(\frac{S}{K}\right) - q\tau .$$

The put-call parity with dividends is:

$$C - P = e^{-q\tau}S - e^{-r\tau}K$$

Stock Index Options

A stock index can be approximated as an asset with continuous dividend yield q.

Examples of exchange traded index options:
S&P 500 (symbol - SPX) – European options
S&P 100 (symbol - OEX) – American options
Options on Dow Jones, NASDAQ 100, etc.

Stock index puts are used for portfolio insurance. An alternative is to create dynamic puts (dynamic portfolio insurance) by executing the dynamic replicating strategy with stock index futures.

FX Options Pricing Formulas

Suppose r_d and r_f are domestic and foreign risk-free interest rates, respectively. Foreign currency can be viewed as a risky asset priced in the units of domestic currency that pays continuous dividends with the dividend yield r_f. Then the pricing formulas for European FX options are (in this case $r = r_d$ and $q = r_f$):

$$C = e^{-r_f\tau}SN(d_+) - e^{-r_d\tau}KN(d_-) ,$$

$$P = e^{-r_d\tau}KN(-d_-) - e^{-r_f\tau}SN(-d_+) ,$$

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + \mu\tau}{\sigma\sqrt{\tau}}, \quad d_+ = d_- + \sigma\sqrt{\tau} , \qquad \mu = r_d - r_f - \frac{\sigma^2}{2} .$$